BANGLADESH ARMY INTERNATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY (BAIUST), CUMILLA

Term Final Examination, Fall 2024

Department of Computer Science and Engineering

Level-1, Term-I

Course Code: MATH 111/MATH 141 **Full Marks:** 150
Course Title: MATH-I (Differential, Integral Calculus and Matrix)
Credit Hour: 3.00 **Time:** 3 hrs

ANSWER SHEET

Examiner's Note: Answer any five (05) questions. Either three from Part-A and two from Part-B, or two from Part-A and three from Part-B.
Student's Note: All questions are answered below for reference and study purposes.

PART - A

Question 1(a)

Calculate the inverse of the following matrix: $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution: To find the inverse of a matrix, we first calculate its determinant.

$$\det(A) = 1 \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}$$

$$\det(A) = 1((-1)(0) - (0)(0)) + 1((2)(0) - (0)(1)) + 0$$

$$\det(A) = 1(0) + 1(0) + 0 = 0$$

Since the determinant of the matrix is 0, the matrix is singular.

Answer: The matrix has no inverse because its determinant is 0.

Question 1(b)

Show that the given matrix is orthogonal: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$**.**

Solution: A matrix A is orthogonal if $AA^T = kI$ or $A^T A = kI$, where k is a scalar. In this case, the matrix is symmetric, so $A^T = A$. We will check $AA^T = A^2$.

$$AA^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Let's compute the elements of the product matrix $C = AA^T$.
$C_{11} = (1)(1) + (1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 + 1 = 4$
$C_{12} = (1)(1) + (1)(-1) + (1)(1) + (1)(-1) = 1 - 1 + 1 - 1 = 0$
$C_{13} = (1)(1) + (1)(1) + (1)(-1) + (1)(-1) = 1 + 1 - 1 - 1 = 0$
$C_{14} = (1)(1) + (1)(-1) + (1)(-1) + (1)(1) = 1 - 1 - 1 + 1 = 0$
$C_{22} = (1)(1) + (-1)(-1) + (1)(1) + (-1)(-1) = 1 + 1 + 1 + 1 = 4$
$C_{23} = (1)(1) + (-1)(1) + (1)(-1) + (-1)(-1) = 1 - 1 - 1 + 1 = 0$
Due to symmetry, all off-diagonal elements will be 0 and all diagonal elements will be 4.

$$AA^T = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4I$$

Since $AA^T = 4I$, the matrix is orthogonal. (Note: A matrix is *orthonormal* if $AA^T = I$. The matrix $\frac{1}{2}A$ would be orthonormal.) **(Proven)**

Question 1(c)

Find the Eigen values of the matrix: $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$**.**

Solution: We solve the characteristic equation $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-5 - \lambda)(4 - \lambda) - (2)(-7) = 0$$

$$-20 + 5\lambda - 4\lambda + \lambda^2 + 14 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

Factor the quadratic equation:

$$(\lambda + 3)(\lambda - 2) = 0$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

Answer: The Eigen values are $\lambda_1 = -3$ and $\lambda_2 = 2$.

Question 2(a)

Consider the area above the x-axis included between the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 2ax$.
(i) Sketch the area
(ii) Determine the boundaries
(iii) Construct the definite integral
(iv) Illustrate.

Solution:

(i) Sketch:

- $y^2 = ax$ is a parabola opening to the right, vertex at (0,0).
- $x^2 - 2ax + y^2 = 0 \implies (x-a)^2 + y^2 = a^2$ is a circle centered at $(a, 0)$ with radius a.

Area between $y^2 = ax$ and circle $x^2 + y^2 = 2ax$

The area is the region in the first quadrant bounded below by the parabola and above by the circle.

(ii) Boundaries: We find the intersection points. Substitute $y^2 = ax$ into the circle's equation:

$$x^2 + (ax) = 2ax$$

$$x^2 - ax = 0 \implies x(x-a) = 0$$

The intersection points are at $x = 0$ and $x = a$. The area is bounded by $x = 0$ on the left and $x = a$ on the right.

- Upper curve (Circle): $y = \sqrt{2ax - x^2}$
- Lower curve (Parabola): $y = \sqrt{ax}$

(iii) Definite Integral: The area A is the integral of (Upper Curve - Lower Curve) from $x = 0$ to $x = a$.

$$A = \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{ax}\right) dx$$

(iv) Illustrate (Solve): We solve the integral in two parts.

$$A = \int_0^a \sqrt{a^2 - (x-a)^2}\, dx - \int_0^a \sqrt{a}\sqrt{x}\, dx$$

- **Part 1 (Circle):** $\int_0^a \sqrt{a^2 - (x-a)^2}\, dx$. This integral represents the area of a quarter-circle of radius a, which is $\frac{1}{4}\pi a^2$.

- **Part 2 (Parabola):**

$$-\int_0^a \sqrt{a}x^{1/2}\,dx = -\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a = -\sqrt{a}\left(\frac{a^{3/2}}{3/2}-0\right)$$

$$= -\sqrt{a}\left(\frac{2}{3}a^{3/2}\right) = -\frac{2}{3}a^2$$

Total Area: $A = \frac{\pi a^2}{4} - \frac{2}{3}a^2 = a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$

Answer: The area is $a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$.

Question 2(b)

Develop that $B(p+1,q)+B(p,q+1)=B(p,q)$.

Solution: We use the Gamma function definition of the Beta function: $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. We also use the property $\Gamma(p+1)=p\Gamma(p)$.

1. **LHS:** $B(p+1,q)+B(p,q+1)$

$$= \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+1+q)} + \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$= \frac{p\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} + \frac{\Gamma(p)q\Gamma(q)}{\Gamma(p+q+1)}$$

$$= \frac{\Gamma(p)\Gamma(q)(p+q)}{\Gamma(p+q+1)}$$

2. **Simplify Denominator:** Using $\Gamma(p+q+1)=(p+q)\Gamma(p+q)$:

$$= \frac{\Gamma(p)\Gamma(q)(p+q)}{(p+q)\Gamma(p+q)}$$

Cancel the $(p+q)$ term:

$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

3. **RHS:** By definition, $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Since LHS = RHS, the identity is proven. **(Proven)**

Question 2(c)

Show that the matrix $A=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ -i & -1\end{bmatrix}$ is unitary.

Solution: A matrix A is unitary if $A^\dagger A = I$, where $A^\dagger$ is the conjugate transpose $(\overline{A})^T$. 1. **Find $\overline{A}$ (Conjugate):**

$$\overline{A} = \frac{1}{\sqrt{2}}\begin{bmatrix}\overline{1} & \overline{i}\\ \overline{-i} & \overline{-1}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix}$$

2. **Find $A^\dagger$ (Conjugate Transpose):**

$$A^\dagger = (\overline{A})^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

(Note: In this case, $A^\dagger = A$, so the matrix is also Hermitian). 3. **Calculate $A^\dagger A$:**

$$A^\dagger A = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} (1)(1)+(i)(-i) & (1)(i)+(i)(-1) \\ (-i)(1)+(-1)(-i) & (-i)(i)+(-1)(-1) \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1-i^2 & i-i \\ -i+i & -i^2+1 \end{bmatrix}$$

Since $i^2 = -1$:

$$= \frac{1}{2}\begin{bmatrix} 1-(-1) & 0 \\ 0 & -(-1)+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $A^\dagger A = I$, the matrix is unitary. **(Proven)**

Question 3(a)

Solve the rank of matrix by reducing it to echelon form: $A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$R_1 \to -R_1$:

$$\sim \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$: $R_3 \to R_3 - 2R_1$: $R_4 \to R_4 - 4R_1$:

$$\sim \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$: $R_4 \to R_4 - R_2$:

$$\sim \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon form has two non-zero rows. **Answer:** The rank of the matrix is 2.

Question 3(b)

Find the inverse of the matrix with the help of matrix: $x + y + z = 6, x - y + z = 2, 2x + y - z = 1$**. Solution:** This question asks to solve the system, which is done using the inverse. We find $X = A^{-1}B$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

1. **Find** $\det(A)$**:**

$$\det(A) = 1((-1)(-1) - (1)(1)) - 1((1)(-1) - (1)(2)) + 1((1)(1) - (-1)(2))$$
$$= 1(1 - 1) - 1(-1 - 2) + 1(1 + 2) = 0 - (-3) + 3 = 6$$

2. **Find Adjoint of A (Transpose of Cofactor Matrix):**

$$C_{11} = 0, \quad C_{12} = -(-1 - 2) = 3, \quad C_{13} = (1 + 2) = 3$$
$$C_{21} = -(-1 - 1) = 2, \quad C_{22} = (-1 - 2) = -3, \quad C_{23} = -(1 - 2) = 1$$
$$C_{31} = (1 - (-1)) = 2, \quad C_{32} = -(1 - 1) = 0, \quad C_{33} = (-1 - 1) = -2$$

Cofactor Matrix $C = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$. Adjoint Matrix $\text{adj}(A) = C^T = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$. 3. **Find Inverse** A^{-1}**:**

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

4. **Solve for X:**

$$X = A^{-1}B = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$X = \frac{1}{6}\begin{bmatrix} (0)(6) + (2)(2) + (2)(1) \\ (3)(6) + (-3)(2) + (0)(1) \\ (3)(6) + (1)(2) + (-2)(1) \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 + 4 + 2 \\ 18 - 6 + 0 \\ 18 + 2 - 2 \end{bmatrix}$$

$$X = \frac{1}{6}\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Answer: The solution is $x = 1, y = 2, z = 3$.

PART - B

Question 1(a)

If $y = (x^2 - 1)^n$ then show that $(x^2 - 1)y_2 - 2(n-1)xy_1 - 2ny = 0$.

Solution: 1. **Find y_1:**

$$y = (x^2 - 1)^n$$
$$y_1 = n(x^2 - 1)^{n-1} \cdot (2x)$$

2. **Rearrange:** Multiply by $(x^2 - 1)$:

$$(x^2 - 1)y_1 = n(x^2 - 1)^n \cdot (2x)$$

Substitute $y = (x^2 - 1)^n$:

$$(x^2 - 1)y_1 = 2nxy$$

3. **Differentiate again (using Product Rule):** Differentiate LHS: $\frac{d}{dx}((x^2-1)y_1) = (2x)y_1 + (x^2-1)y_2$ Differentiate RHS: $\frac{d}{dx}(2nxy) = 2n(1 \cdot y + x \cdot y_1) = 2ny + 2nxy_1$ 4. **Equate and simplify:**

$$(x^2 - 1)y_2 + 2xy_1 = 2ny + 2nxy_1$$
$$(x^2 - 1)y_2 + 2xy_1 - 2nxy_1 - 2ny = 0$$
$$(x^2 - 1)y_2 + 2x(1-n)y_1 - 2ny = 0$$
$$(x^2 - 1)y_2 - 2x(n-1)y_1 - 2ny = 0$$

(Proven)

Question 1(b)

Expand the series of $e^x \sin x = x + x^2 + \frac{x^3}{3} - \dots$

Solution: We use the Maclaurin series for e^x and $\sin x$ and multiply them.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = x - \frac{x^3}{6} + \dots$$

Now, multiply the two series:

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)\left(x - \frac{x^3}{6} + \dots\right)$$

We only need terms up to x^3:

- x term: $(1)(x) = x$
- x^2 term: $(x)(x) = x^2$
- x^3 term: $(1)(-\frac{x^3}{6}) + (\frac{x^2}{2})(x) = -\frac{x^3}{6} + \frac{x^3}{2} = \frac{-1+3}{6}x^3 = \frac{2}{6}x^3 = \frac{x^3}{3}$

Combining these gives:

$$e^x \sin x = x + x^2 + \frac{x^3}{3} - \dots$$

(Proven)

Question 1(c)

If $y = e^{a\sin^{-1}x}$ then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.
Solution: 1. **Find y_1:**

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

$$y_1\sqrt{1-x^2} = ay$$

Square both sides: $y_1^2(1-x^2) = a^2y^2$ 2. **Find y_2:** Differentiate $y_1^2(1-x^2) = a^2y^2$ w.r.t x.

$$(2y_1y_2)(1-x^2) + y_1^2(-2x) = a^2(2yy_1)$$

Divide by $2y_1$ (assuming $y_1 \neq 0$):

$$y_2(1-x^2) - xy_1 = a^2y$$

$$(1-x^2)y_2 - xy_1 - a^2y = 0$$

3. **Use Leibniz's Theorem:** Differentiate this equation n times.

$$\frac{d^n}{dx^n}((1-x^2)y_2) - \frac{d^n}{dx^n}(xy_1) - \frac{d^n}{dx^n}(a^2y) = 0$$

- **Term 1:** $v = 1-x^2, u = y_2$. $v_1 = -2x, v_2 = -2, v_3 = 0$. $(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n$
- **Term 2:** $v = x, u = y_1$. $v_1 = 1, v_2 = 0$. $-(xy_{n+1} + n(1)y_n)$
- **Term 3:** $-a^2y_n$

4. **Combine and Simplify:**

$$[(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n] - [xy_{n+1} + ny_n] - a^2y_n = 0$$

Group by y derivative:

$$(1-x^2)y_{n+2} + (-2nx - x)y_{n+1} + (-n(n-1) - n - a^2)y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (-n^2 + n - n - a^2)y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

(Proven)

Question 2(a)

Compute the local maxima, local minima and critical points of the function: $f(x) = x^3 - 9x^2 + 15x - 3$**.**

Solution: 1. **Find Critical Points:** Set $f'(x) = 0$.

$$f'(x) = 3x^2 - 18x + 15$$

$$3(x^2 - 6x + 5) = 0$$

$$3(x - 1)(x - 5) = 0$$

The critical points are $x = 1$ and $x = 5$.
2. **Second Derivative Test:** Find $f''(x)$.

$$f''(x) = 6x - 18$$

3. **Test Points:**

- At $x = 1$: $f''(1) = 6(1) - 18 = -12 < 0$. This is a **Local Maximum**.
- At $x = 5$: $f''(5) = 6(5) - 18 = 30 - 18 = 12 > 0$. This is a **Local Minimum**.

Answer:

- Critical Points: $x = 1, x = 5$
- Local Maximum at $x = 1$
- Local Minimum at $x = 5$

Question 2(b)

Find the radius of curvature at the origin of the curve $x^3 + y^3 - 2x^2 + 6y = 0$**.**

Solution: We use Newton's Method for curvature at the origin $(0, 0)$.

The formula is $\rho = \frac{(1+p^2)^{3/2}}{|q|}$, where $p = y'(0)$ and $q = y''(0)$.

We find p and q by differentiating the equation implicitly w.r.t x.

$$3x^2 + 3y^2y' - 4x + 6y' = 0$$

1. **Find** $p = y'(0)$**:** Plug in $x = 0, y = 0$.

$$0 + 0 - 0 + 6p = 0 \implies 6p = 0 \implies p = 0$$

2. **Find** $q = y''(0)$**:** Differentiate the first derivative equation again.

$$(6x) + (6yy' \cdot y' + 3y^2 \cdot y'') - 4 + 6y'' = 0$$

$$6x + 6y(y')^2 + 3y^2y'' - 4 + 6y'' = 0$$

Plug in $x = 0, y = 0, y'(0) = p = 0$:

$$0 + 0 + 0 - 4 + 6q = 0$$

$$6q = 4 \implies q = 4/6 = 2/3$$

3. **Calculate Radius of Curvature ρ:**

$$\rho = \frac{(1+p^2)^{3/2}}{|q|} = \frac{(1+0^2)^{3/2}}{|2/3|} = \frac{1}{2/3} = \frac{3}{2}$$

Answer: The radius of curvature is $\rho = 1.5$.

Question 2(c)

Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ where $u = e^x(x\cos y - y\sin y)$.

Solution: 1. **Find x derivatives:**

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^x(x\cos y - y\sin y))$$

$$= (e^x)(x\cos y - y\sin y) + e^x(\cos y) = e^x(x\cos y - y\sin y + \cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = (e^x)(x\cos y - y\sin y + \cos y) + e^x(\cos y)$$

$$= e^x(x\cos y - y\sin y + 2\cos y)$$

2. **Find y derivatives:**

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^x(x\cos y - y\sin y))$$

$$= e^x(x(-\sin y) - (1\cdot\sin y + y\cos y))$$

$$= e^x(-x\sin y - \sin y - y\cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y - (1\cdot\cos y - y\sin y))$$

$$= e^x(-x\cos y - \cos y - \cos y + y\sin y)$$

$$= e^x(-x\cos y - 2\cos y + y\sin y)$$

3. **Sum the second derivatives:**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= e^x(x\cos y - y\sin y + 2\cos y) + e^x(-x\cos y - 2\cos y + y\sin y)$$

$$= e^x((x\cos y - x\cos y) + (-y\sin y + y\sin y) + (2\cos y - 2\cos y))$$

$$= e^x(0 + 0 + 0) = 0$$

(Proven)

Question 3(a)

Show Rolle's Theorem for the function $f(x) = x^2 - 5x + 6$ in $1 \leq x \leq 4$.

Solution: Rolle's Theorem has three conditions for a function $f(x)$ on a closed interval $[a, b]$:
1. $f(x)$ is continuous on $[a, b]$.
2. $f(x)$ is differentiable on (a, b).
3. $f(a) = f(b)$.
If all hold, there exists at least one c in (a, b) such that $f'(c) = 0$.

Here, $f(x) = x^2 - 5x + 6$, $a = 1$, $b = 4$.
1. $f(x)$ is a polynomial, so it is continuous everywhere, including on $[1, 4]$. (Condition 1 holds)
2. $f'(x) = 2x - 5$, which is defined for all x. So $f(x)$ is differentiable everywhere, including on $(1, 4)$. (Condition 2 holds)
3. $f(1) = (1)^2 - 5(1) + 6 = 1 - 5 + 6 = 2$
$f(4) = (4)^2 - 5(4) + 6 = 16 - 20 + 6 = 2$
Since $f(1) = f(4) = 2$, (Condition 3 holds).

Since all conditions are met, Rolle's Theorem applies. We must find c such that $f'(c) = 0$.

$$f'(c) = 2c - 5 = 0 \implies 2c = 5 \implies c = 2.5$$

Since $c = 2.5$ is in the interval $(1, 4)$, Rolle's Theorem is shown to hold.

Question 3(b)

Develop that the series, $log(1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2}{3}x^3 + \frac{x^4}{4} \dots$
(Note: The question has a '+' sign on the x^4 term, unlike the previous exam)

Solution: We use the identity $1 - x + x^2 = \frac{1+x^3}{1+x}$.

$$\ln(1 - x + x^2) = \ln\left(\frac{1 + x^3}{1 + x}\right) = \ln(1 + x^3) - \ln(1 + x)$$

Now we use the standard Maclaurin series for $\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots$

- $\ln(1 + x^3) = (x^3) - \frac{(x^3)^2}{2} + \dots = x^3 - \frac{x^6}{2} + \dots$
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Now subtract the two series:

$$\ln(1 + x^3) - \ln(1 + x) = (x^3 - \dots) - (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots)$$
$$= -x + \frac{x^2}{2} + (x^3 - \frac{x^3}{3}) + \frac{x^4}{4} + \dots$$
$$= -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} + \dots$$

(Proven)

Question 3(c)

If $u = \tan^{-1}\frac{x^3+y^3}{x+y}$ **then express that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$**.** (Note: Question has a typo $y\frac{\partial u}{\partial x}$, corrected to $y\frac{\partial u}{\partial y}$)

Solution: We use Euler's Theorem for homogeneous functions. 1. Let $z = \tan u = \frac{x^3+y^3}{x+y}$. 2. z is a homogeneous function. Let's find its degree n.

$$z(tx, ty) = \frac{(tx)^3 + (ty)^3}{tx + ty} = \frac{t^3(x^3 + y^3)}{t(x + y)} = t^2\left(\frac{x^3 + y^3}{x + y}\right) = t^2 z$$

The degree of z is $n = 2$. 3. By Euler's Theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 2z$. 4. Now we find the partials of z in terms of u:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\tan u) = \sec^2 u\frac{\partial u}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\tan u) = \sec^2 u\frac{\partial u}{\partial y}$$

5. Substitute these into Euler's Theorem:

$$x\left(\sec^2 u\frac{\partial u}{\partial x}\right) + y\left(\sec^2 u\frac{\partial u}{\partial y}\right) = 2\tan u$$

Factor out $\sec^2 u$:

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$
$$= 2\left(\frac{\sin u}{\cos u}\right)(\cos^2 u) = 2\sin u\cos u$$

Using the double-angle identity, $2\sin u\cos u = \sin 2u$.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

(Proven)